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Biorthogonal Two-direction Wavelet Packets with a Positive Integer Dilation Factor*

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Abstract: Biorthogonal two-direction wavelet packets with dilation factor are introduced and their properties are discussed by means of the matrix theory and operator theory. A new approach for constructing biorthogonal two-direction wavelet packets is developed. The formulae for performing iteration and decomposition are established. New Riesz bases for $L^2(\mathbb{R})$ are obtained by the given biorthogonal two-direction wavelet packets. Finally, an example for constructing biorthogonal two-direction wavelet packets is given.

Keywords: two-direction wavelet; two-direction wavelet packets; two-direction refinable function; biorthogonal

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1 Introduction

Wavelet transform is an important mathematical tool with which data or functions can be divided into different frequency components. It is well known that there is a limitation for the time-frequency localization of a single wavelet. In other words, an orthogonal wavelet function with compact support and certain regularity is not symmetric[1,2]. Therefore, Geronimo *et al*[3] constructed two functions $\psi_1(x)$ and $\psi_2(x)$ whose translations and dilations form an orthonormal basis in $L^2(\mathbb{R})$. The importance for these two functions is that they are continuous, well time-localized (or short support), and symmetry. This example tells us that if multiwavelets are used in an expansion, then better properties can be achieved.

Orthogonal wavelet packets were firstly introduced by Coifman and Meyer[4]. Orthogonal wavelet packets are used to further decompose wavelet components. Wavelet packets, due to their nice characteristics, have been widely applied to signal processing[5], image processing[6] and so on. Biorthogonal wavelet packets were given by Daubechies and Cohen[7]. Biorthogonal wavelet packets are more flexible in application. In addition, wavelet packets provide better frequency localization than wavelets while time-domain localization is not lost.

The two-scale refinable equation with scale $\alpha\,(2 \le \alpha \in \mathbb{Z})$

$$\phi(t) = \sum_{u\in\mathbb{Z}} p_u \phi(\alpha t - u) \tag{1}$$

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plays a basic role in the construction and application of scalar wavelets[8,9]. Yang[10] generalized the two-scale refinable equation, and established the biorthogonality criteria for two-direction refinable function and two-direction wavelets. Motivated by [10,11], we give the definition of biorthogonal two-direction wavelet packets and discuss their properties by means of matrix theory. The formulae for performing iterations and decomposition are also established.

2 Two-direction multiresolution analysis with dilation α

We begin with recalling some basic notations and results used later. Let $\mathbb{R}$ and $\mathbb{C}$ be the sets of all real and complex numbers, respectively. Denote by $\mathbb{Z}$ and $\mathbb{Z}_+$ the set of all integers and nonnegative integers, respectively. To obtain a uniform method for constructing biorthogonal two-direction wavelet with dilation α, let us give two-direction multiresolution analysis.

Define the operators R, T and D as follows:

$$(Rf)(t)=f(-t),\quad (Tf)(t)=f(t-1),\quad (D_\alpha f)t=\alpha^{\frac{1}{2}}f(\alpha t),\quad \forall\, f\in L^2(\mathbb{R}),\quad t\in\mathbb{R}.$$

Then R, T and D are unitary operators on the Hilbert space $L^2(\mathbb{R})$.

Definition 2.1 Given $2\le \alpha\in\mathbb{Z}$ and $\phi\in L^2(\mathbb{R})$. If there exist $\{p_u^+\}_{u\in\mathbb{Z}}\in\ell^2(\mathbb{Z})$ and $\{p_u^-\}_{u\in\mathbb{Z}}\in\ell^2(\mathbb{Z})$ such that

$$\phi=\sum_{u\in\mathbb{Z}}p_u^+\alpha^{-1/2}D_\alpha T^u\phi+\sum_{u\in\mathbb{Z}}p_u^-\alpha^{-1/2}D_\alpha T^{-u}R\phi, \tag{2}$$

then ϕ is said to be a two-direction refinable function (TDRF). The sequences $\{p_u^+\}_{u\in\mathbb{Z}}$ and $\{p_u^-\}_{u\in\mathbb{Z}}$ are called the positive-direction mask (PDM) and negative-direction mask (NDM) of ϕ, respectively.

The equation (2) can be simply written as

$$\phi(t)=\sum_{u\in\mathbb{Z}}p_u^+\phi(\alpha t-u)+\sum_{u\in\mathbb{Z}}p_u^-\phi(u-\alpha t), \tag{3}$$

which becomes two-scale equation (1) in the case of $p_u^-=0$.

By taking the Fourier transformation for the both sides of (2), we have

$$\hat\phi(\omega)=p^+(\mathrm{e}^{\frac{-\mathrm{i}\omega}{\alpha}})\,\hat\phi(\omega/\alpha)+p^-(\mathrm{e}^{\frac{-\mathrm{i}\omega}{\alpha}})\,\overline{\hat\phi(\omega/\alpha)}, \tag{4}$$

where

$$p^+(z)=(1/\alpha)\sum_{u\in\mathbb{Z}}p_u^+z^u,\quad z=\mathrm{e}^{\frac{-\mathrm{i}\omega}{\alpha}}$$

is called the positive-direction mask symbol (PDMS) and

$$p^-(z)=\frac{1}{\alpha}\sum_{u\in\mathbb{Z}}p_u^-z^u$$

is called the negative-direction mask symbol (NDMS).

Proposition 2.1 Let ϕ be a TDRF. If there exist sequences $\{p_u^+\}_{u\in\mathbb{Z}}$ and $\{p_u^-\}_{u\in\mathbb{Z}}$ of ϕ, then ϕ satisfies the following equation

$$\widehat\Phi(\omega):=\begin{bmatrix}\overline{\hat\phi(\omega)}\\ \hat\phi(\omega)\end{bmatrix}=\begin{bmatrix}\overline{p^+(\mathrm{e}^{\frac{-\mathrm{i}\omega}{\alpha}})} & \overline{p^-(\mathrm{e}^{\frac{-\mathrm{i}\omega}{\alpha}})}\\ p^+(\mathrm{e}^{\frac{-\mathrm{i}\omega}{\alpha}}) & p^-(\mathrm{e}^{\frac{-\mathrm{i}\omega}{\alpha}})\end{bmatrix}\begin{bmatrix}\overline{\hat\phi(\omega/\alpha)}\\ \hat\phi(\omega/\alpha)\end{bmatrix}, \tag{5}$$

where $p^+(z)$ is the PDMS and $p^-(z)$ is the NDMS.

Proof By taking the Fourier translation, we obtain (4). On the other hand, we rewrite (2) as

$$R\phi = \sum_{u\in\mathbb{Z}} p_u^+ \alpha^{-1/2} D_\alpha T^u R\phi + \sum_{u\in\mathbb{Z}} p_u^- \alpha^{-1/2} D_\alpha T^{-u}\phi. \tag{6}$$

Simply

$$\phi(-t) = \sum_{u\in\mathbb{Z}} p_u^+ \phi(-\alpha t - u) + \sum_{u\in\mathbb{Z}} p_u^- \phi(u + \alpha t). \tag{7}$$

Also, by implementing the Fourier transformation for the both sides of (6), we have

$$\overline{\hat{\phi}(\omega)} = \overline{p^+(\mathrm{e}^{\frac{-\mathrm{i}\omega}{\alpha}})}\,\overline{\hat{\phi}(\omega/\alpha)} + \overline{p^-(\mathrm{e}^{\frac{-\mathrm{i}\omega}{\alpha}})}\,\hat{\phi}(\omega/\alpha). \tag{8}$$

(4) and (8) can be written as (5). The proof is completed.

Let ϕ be a two-direction refinable function with masks $\{p_u^+\}_{u\in\mathbb{Z}}$ and $\{p_u^-\}_{u\in\mathbb{Z}}$. By virtue of the positive-direction mask $\{p_u^+\}_{u\in\mathbb{Z}}$ and the negative-direction mask $\{p_u^-\}_{u\in\mathbb{Z}}$, we construct the following matrix equation

$$\Phi(t) = \begin{bmatrix} \phi(-t) \\ \phi(t) \end{bmatrix} = \sum_{u\in\mathbb{Z}} \begin{bmatrix} p_{-u}^- & p_{-u}^+ \\ p_u^+ & p_u^- \end{bmatrix} \Phi(\alpha t - u). \tag{9}$$

It is easy to see that (5) and (9) are equivalent. The matrix

$$P(z) = \frac{1}{\alpha}\sum_{u\in\mathbb{Z}} \begin{bmatrix} p_{-u}^- & p_{-u}^+ \\ p_u^+ & p_u^- \end{bmatrix} z^u, \quad z = \mathrm{e}^{\frac{-\mathrm{i}\omega}{\alpha}},$$

is called the matrix mask symbol of $\Phi(t)$.

Definition 2.2 A pair $\{\phi, \widetilde{\phi}\}$ of two-direction refinable functions is said to be biorthogonal if

$$\langle \phi,\, T^k\widetilde{\phi}\rangle = \delta_{0,k}, \quad \langle \phi,\, T^u R\widetilde{\phi}\rangle = 0, \quad \forall\, k,\, u \in \mathbb{Z}, \tag{10}$$

where $\delta_{0,k}$ is the Kronecker symbol.

For a function $\phi \in L^2(\mathbb{R})$, we define a subspace sequence $V_j \in L^2(\mathbb{R})$ by

$$V_j = \bigvee\Big(\{D_\alpha^j T^k\phi\}_{k\in\mathbb{Z}} \bigcup \{D_\alpha^j T^u R\phi\}_{u\in\mathbb{Z}}\Big). \tag{11}$$

Definition 2.3 The sequence $\{V_j\}_{j\in\mathbb{Z}}$ defined by (11) is said to be a two-direction multiresolution analysis (TDMRA) with scale α generated by ϕ if the following conditions are satisfied.

(i) $V_n \subset V_{n+1}$, for all $n \in \mathbb{Z}$;

(ii) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$, $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$;

(iii) $f \in V_0$ if and only if $D_\alpha^j f \in V_j$, for all $j \in \mathbb{Z}$;

(iv) The sequence $\{T^k\phi\}_{k\in\mathbb{Z}} \bigcup \{T^n R\phi\}_{n\in\mathbb{Z}}$ is a Riesz basis of V_0.

Let $\Lambda = \{1, 2, \cdots, \alpha - 1\}$.

Definition 2.4 Let $\{V_j\}_{j\in\mathbb{Z}}$ be a TDMRA generated by ϕ. A subset $\{\psi_\iota : \iota \in \Lambda\}$ of $L^2(\mathbb{R})$ is called a two-direction wavelet set (TDWS) associated to ϕ with scale α if

$$\{T^k\psi_\iota\}_{k\in\mathbb{Z}} \bigcup \{T^u R\psi_\iota\}_{u\in\mathbb{Z}}, \quad \iota \in \lambda,$$

forms a Riesz basis for W_0 where $W_0 = V_1 \ominus V_0$.

Proposition 2.2 Let $\{\psi_\iota : \iota \in \Lambda\}$ of $L^2(\mathbb{R})$ be a TDWS associated to ϕ with scale α. If there exist sequences $\{q^+_{u,\iota}\}_{u\in\mathbb{Z}}$ and $\{q^-_{u,\iota}\}_{u\in\mathbb{Z}}$ of $\{\psi_\iota : \iota \in \Lambda\}$, then $\{\psi_\iota : \iota \in \Lambda\}$ satisfies the following equation

$$\widehat{\Psi}_\iota(\omega) = \begin{bmatrix} \overline{\hat{\psi}_\iota(\omega)} \\ \hat{\psi}_\iota(\omega) \end{bmatrix} = \begin{bmatrix} \overline{q^-_\iota(\mathrm{e}^{\frac{-\mathrm{i}\omega}{\alpha}})} & \overline{q^+_\iota(\mathrm{e}^{\frac{-\mathrm{i}\omega}{\alpha}})} \\ q^+_\iota(\mathrm{e}^{\frac{-\mathrm{i}\omega}{\alpha}}) & q^-_\iota(\mathrm{e}^{\frac{-\mathrm{i}\omega}{\alpha}}) \end{bmatrix} \begin{bmatrix} \overline{\hat{\phi}(\frac{\omega}{\alpha})} \\ \hat{\phi}(\frac{\omega}{\alpha}) \end{bmatrix}, \quad \iota \in \Lambda, \tag{12}$$

where

$$q^+_\iota(z) = (1/\alpha)\sum_{u\in\mathbb{Z}} q^+_{u,\iota} z^u, \quad q^-_\iota(z) = \frac{1}{\alpha}\sum_{u\in\mathbb{Z}} q^-_{u,\iota} z^u, \quad z = \mathrm{e}^{\frac{-\mathrm{i}\omega}{\alpha}}.$$

Proof For $\{\psi_\iota : \iota \in \Lambda\}$, we have

$$\psi_\iota = \sum_{u\in\mathbb{Z}} q^+_{u,\iota}\alpha^{-1/2} D_\alpha T^u \phi + \sum_{u\in\mathbb{Z}} q^-_{u,\iota}\alpha^{-1/2} D_\alpha T^{-u} R\phi, \quad \iota \in \Lambda. \tag{13}$$

This equation can be simply written as

$$\psi_\iota(t) = \sum_{u\in\mathbb{Z}} q^+_{u,\iota}\phi(\alpha t - u) + \sum_{u\in\mathbb{Z}} q^-_{u,\iota}\phi(u - \alpha t), \quad \iota \in \Lambda. \tag{14}$$

Implementing the Fourier transformation for the both sides of (13) yields

$$\widehat{\psi}_\iota(\omega) = q^+_\iota(\mathrm{e}^{\frac{-\mathrm{i}\omega}{\alpha}})\hat{\phi}(\omega/\alpha) + q^-_\iota(\mathrm{e}^{\frac{-\mathrm{i}\omega}{\alpha}})\,\overline{\hat{\phi}(\omega/\alpha)}, \quad \iota \in \Lambda. \tag{15}$$

We rewrite equation (14) as

$$\psi_\iota(-t) = \sum_{u\in\mathbb{Z}} q^+_{u,\iota}\phi(-\alpha t - u) + \sum_{u\in\mathbb{Z}} q^-_{u,\iota}\phi(u + \alpha t), \quad \iota \in \Lambda. \tag{16}$$

The refinement equations (14) and (16) lead to the following formula

$$\Psi_\iota(t) = \begin{bmatrix} \psi_\iota(-t) \\ \psi_\iota(t) \end{bmatrix} = \sum_{u\in\mathbb{Z}} \begin{bmatrix} q^-_{-u,\iota} & q^+_{-u,\iota} \\ q^+_{u,\iota} & q^-_{u,\iota} \end{bmatrix} \Phi(\alpha t - u), \quad \iota \in \Lambda. \tag{17}$$

The frequency field form of the relation formula (17) is (12). The proof is completed.

3 Two-direction wavelet packets

We shall introduce the two-direction wavelet packets and investigate their properties. Let ϕ and $\widetilde{\phi}$ be a pair of biorthogonal two-direction refinable functions, we rewrite the symbols as

$$\psi_0(t) = \phi(t), \quad p^{+(0)}_u = p^+_u, \quad p^{-(0)}_u = p^-_u, \quad p^{+(\iota)}_u = q^+_{u,\iota}, \quad p^{-(\iota)}_u = q^-_{u,\iota}, \quad \iota \in \Lambda, \quad u \in \mathbb{Z},$$

$$\widetilde{\psi}_0(t) = \widetilde{\phi}(t), \quad \tilde{p}^{+(0)}_u = \tilde{p}^+_u, \quad \tilde{p}^{-(0)}_u = \tilde{p}^-_u, \quad \tilde{p}^{+(\iota)}_u = \tilde{q}^+_{u,\iota}, \quad \tilde{p}^{-(\iota)}_u = \tilde{q}^-_{u,\iota}, \quad \iota \in \Lambda, \quad u \in \mathbb{Z},$$

$$P^{(\lambda)}(z) = \frac{1}{\alpha}\sum_{u\in\mathbb{Z}} \begin{bmatrix} p^{+(\lambda)}_u & p^{-(\lambda)}_u \\ p^{-(\lambda)}_{-u} & p^{+(\lambda)}_{-u} \end{bmatrix} z^u, \quad \tilde{P}^{(\lambda)}(z) = \frac{1}{\alpha}\sum_{u\in\mathbb{Z}} \begin{bmatrix} \tilde{p}^{+(\lambda)}_u & \tilde{p}^{-(\lambda)}_u \\ \tilde{p}^{-(\lambda)}_{-u} & \tilde{p}^{+(\lambda)}_{-u} \end{bmatrix} z^u, \quad z = \mathrm{e}^{-\mathrm{i}\omega/\alpha}, \quad \iota \in \Lambda.$$

Definition 3.1 Two families of two-direction refinable functions $\{\psi_{\alpha n+\lambda}(t), n \in \mathbb{Z}^+, \lambda = 0, 1, \cdots, \alpha-1\}$ and $\{\tilde{\psi}_{\alpha n+\lambda}(t), n \in \mathbb{Z}^+, \lambda = 0, 1, \cdots, \alpha-1\}$ are called biorthogonal two-direction wavelet packets (TDWPs) with respect to a pair of biorthogonal two-direction scaling functions $\phi(t)$ and $\tilde{\phi}(t)$, respectively, if

$$\psi_{\alpha n+\lambda}(t) = \sum_{u\in\mathbb{Z}} p_u^{+(\lambda)} \psi_n(\alpha t - u) + \sum_{u\in\mathbb{Z}} p_u^{-(\lambda)} \psi_n(u - \alpha t), \tag{18}$$

$$\tilde{\psi}_{\alpha n+\lambda}(t) = \sum_{u\in\mathbb{Z}} \tilde{p}_u^{+(\lambda)} \tilde{\psi}_n(\alpha t - u) + \sum_{u\in\mathbb{Z}} \tilde{p}_u^{-(\lambda)} \tilde{\psi}_n(u - \alpha t). \tag{19}$$

By implementing the Fourier transformation for the both side of (18) and (19), respectively, we have

$$\hat{\psi}_{\alpha n+\lambda}(\omega) = P^{+(\lambda)} \hat{\psi}_n(\omega/\alpha) + P^{-(\lambda)} \overline{\hat{\psi}_n(\omega/\alpha)}, \quad \lambda = 0, 1, \cdots, \alpha-1, \tag{20}$$

$$\hat{\tilde{\psi}}_{\alpha n+\lambda}(\omega) = \tilde{P}^{+(\lambda)} \hat{\tilde{\psi}}_n(\omega/\alpha) + \tilde{P}^{-(\lambda)} \overline{\hat{\tilde{\psi}}_n(\omega/\alpha)}, \quad \lambda = 0, 1, \cdots, \alpha-1, \tag{21}$$

where

$$P^{+(\lambda)}(\omega) = \frac{1}{\alpha} \sum_{u\in\mathbb{Z}} p_u^{+(\lambda)} \mathrm{e}^{-\mathrm{i}u\omega}, \quad P^{-(\lambda)}(\omega) = \frac{1}{\alpha} \sum_{u\in\mathbb{Z}} p_u^{-(\lambda)} \mathrm{e}^{-\mathrm{i}u\omega}, \quad \lambda = 0, 1, \cdots, \alpha-1. \tag{22}$$

$$\tilde{P}^{+(\lambda)}(\omega) = \frac{1}{\alpha} \sum_{u\in\mathbb{Z}} \tilde{p}_u^{+(\lambda)} \mathrm{e}^{-\mathrm{i}u\omega}, \quad \tilde{P}^{-(\lambda)}(\omega) = \frac{1}{\alpha} \sum_{u\in\mathbb{Z}} \tilde{p}_u^{-(\lambda)} \mathrm{e}^{-\mathrm{i}u\omega}, \quad \lambda = 0, 1, \cdots, \alpha-1. \tag{23}$$

Properties and advantages of the biorthogonal two-direction wavelet packets with dilation α are investigated as follows. By applying the same method, one can obtain orthogonal two-direction packets with dilation α given by (18).

Lemma 3.1 Suppose $\Phi(t)$ and $\tilde{\Phi}(t)$ are biorthogonal scaling function vectors. $P(z)$ and $\tilde{P}(z)$ are their matrix symbols, respectively, and ω_j $(j = 1, 2, \cdots, \alpha)$ are roots of equation $z^\alpha - 1 = 0$. Then

$$\sum_{j=1}^{\alpha} P(\omega_j z) \tilde{P}(\omega_j z)^* = I_r, \quad z^\alpha - 1 = 0. \tag{24}$$

It is equivalent to

$$\sum_{\ell\in\mathbb{Z}} P_\ell \tilde{P}_{\ell+\alpha k} = \alpha \delta_{0,k} I_r, \quad k \in \mathbb{Z}. \tag{25}$$

We can easily prove Theorem 3.1 by applying the lemma.

Theorem 3.1 Suppose that $\psi_\iota(t)$ and $\widetilde{\psi}_\iota(t)$ are biorthogonal two-direction wavelets associated to $\phi(t)$ and $\tilde{\phi}(t)$, respectively, ω_j $(j = 1, 2, \cdots, \alpha)$ are roots of equations $z^\alpha - 1 = 0$ with $z = \mathrm{e}^{-\frac{\mathrm{i}\omega}{\alpha}}$. Then

$$\sum_{j=1}^{\alpha} P^{(\iota)}(\omega_j z) \tilde{P}(\omega_j z)^* = O, \quad \iota \in \Lambda, \tag{26}$$

$$\sum_{j=1}^{\alpha} P(\omega_j z) \tilde{P}^{(\iota)}(\omega_j z)^* = O, \quad \iota \in \Lambda, \tag{27}$$

$$\sum_{j=1}^{\alpha} P^{(\nu)}(\omega_j z) \tilde{P}^{(\nu)}(\omega_j z)^* = I_{(\alpha-1)r}, \quad \iota \in \Lambda. \tag{28}$$

Theorem 3.2 Suppose $n \in \mathbb{Z}$ and

$$n=\sum_{j=1}^{\infty}\beta_j\alpha^{j-1},\quad \beta_j\in\{0,1,2,\cdots,\alpha-1\}. \tag{29}$$

Then

$$\hat{\psi}_n(\omega)=\prod_{j=1}^{\infty}P^{(\beta_j)}(\mathrm{e}^{\frac{-\mathrm{i}\omega}{\alpha^j}})\hat{\psi}_0(0), \tag{30}$$

$$\hat{\tilde{\psi}}_n(\omega)=\prod_{j=1}^{\infty}\tilde{P}^{(\beta_j)}(\mathrm{e}^{\frac{-\mathrm{i}\omega}{\alpha^j}})\hat{\tilde{\psi}}_0(0). \tag{31}$$

Proof When $n=0$, (30) follows from (4). Suppose (30) holds for $0\le n<\alpha^{r_0}$. When $\alpha^{r_0}<n<\alpha^{r_0+1}$, by using (20) and the inductive assumption, we have

$$\begin{aligned}\hat{\psi}_n(\omega) &= P^{(\beta_1)}(\mathrm{e}^{\frac{-\mathrm{i}\omega}{\alpha}})\hat{\psi}_{[\frac{n}{\alpha}]}\\ &= P^{(\beta_1)}(\mathrm{e}^{\frac{-\mathrm{i}\omega}{\alpha}})\prod_{j=1}^{\infty}P^{(\beta_{j+1})}(\mathrm{e}^{\frac{-\mathrm{i}\omega}{\alpha^{j+1}}})\hat{\psi}_0(0)=\prod_{j=1}^{\infty}P^{(\beta_j)}(\mathrm{e}^{\frac{-\mathrm{i}\omega}{\alpha^j}})\hat{\psi}_0(0).\end{aligned}$$

Then (30) holds by induction. We can also obtain (31) by using the same argument. The proof is completed.

Theorem 3.3 Suppose k, $l\in\mathbb{Z}$ and $n\in\mathbb{Z}_+$. Then

$$\langle\psi_n(x-k),\,\tilde{\psi}_n(x-l)\rangle=\delta_{k,l}I_r. \tag{32}$$

Proof When $n=0$, (32) holds. Suppose (32) holds for $0\le n<\alpha^{r_0}$. When $\alpha^{r_0}<n<\alpha^{r_0+1}$, by using the inductive assumption (26), we have

$$\begin{aligned}&\langle\psi_n(t-k),\,\tilde{\psi}_n(t-l)\rangle\\ =&\ \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\psi}_n(\omega)\hat{\tilde{\psi}}_n(\omega)\mathrm{e}^{-\mathrm{i}(k-l)\omega}\mathrm{d}\omega\\ =&\ \frac{1}{2\pi}\sum_{j\in\mathbb{Z}}\int_{2\alpha\pi j}^{2\alpha\pi(j+1)}P^{(\beta_1)}(z)\tilde{P}^{(\beta_1)*}(z)\hat{\psi}_{[\frac{n}{\alpha}]}\hat{\tilde{\psi}}_{[\frac{n}{\alpha}]}\mathrm{e}^{-\mathrm{i}(k-l)\omega}\mathrm{d}\omega\\ =&\ \frac{1}{2\pi}\int_0^{2\alpha\pi}P^{(\beta_1)}(z)\tilde{P}^{(\beta_1)*}(z)\sum_{j\in\mathbb{Z}}\hat{\psi}_{[\frac{n}{\alpha}]}(\omega+2\alpha\pi j)\cdot\hat{\tilde{\Psi}}_{[\frac{n}{\alpha}]}(\omega+2\alpha\pi j)\mathrm{e}^{-\mathrm{i}(k-l)\omega}\mathrm{d}\omega\\ =&\ \frac{1}{2\pi}\int_0^{2\alpha\pi}P^{(\beta_1)}(z)\tilde{P}^{(\beta_1)*}(z)\mathrm{e}^{-\mathrm{i}(k-l)\omega}\mathrm{d}\omega\\ =&\ \frac{1}{2\pi}\int_0^{2\pi}\Big(\sum_{m=0}^{\alpha-1}P^{(\beta_1)}(\omega_m z)\tilde{P}^{(\beta_1)*}(\omega_m z)\Big)\mathrm{e}^{-\mathrm{i}(k-l)\omega}\mathrm{d}\omega\\ =&\ \frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}(k-l)\omega}I_r\,\mathrm{d}\omega=\delta_{k,l}I_r.\end{aligned}$$

Then formula (32) is established by the inductive assumption. The proof is completed.

Theorem 3.4 Suppose $k,\ l \in \mathbb{Z}$ and $n \in \mathbb{Z}$ and $\lambda \in \{1, 2, \cdots, \alpha - 1\}$. We have

$$\big\langle \psi_{\alpha n}(x-k),\, \tilde{\psi}_{\alpha n+\lambda}(x-l)\big\rangle = 0. \tag{33}$$

Proof By applying (18), (23), we have

$$\begin{aligned}
&\big\langle \psi_{\alpha n}(x-k),\, \tilde{\psi}_{\alpha n+\lambda}(x-k)\big\rangle \\
=\ &\frac{1}{2\pi}\int_{-\infty}^{\infty} P^{(0)}(z)\tilde{P}^{(\lambda)*}(z)\hat{\psi}_n(\omega)\hat{\tilde{\psi}}_n(\omega)\mathrm{e}^{-\mathrm{i}(k-l)\omega}\mathrm{d}\omega \\
=\ &\frac{1}{2\pi}\int_{0}^{2\alpha\pi} P^{(0)}(z)\tilde{P}^{(\lambda)*}(z)\sum_{m\in\mathbb{Z}}\hat{\psi}_n(\omega+2m\alpha\pi)\cdot\hat{\tilde{\psi}}_n(\omega+2m\alpha\pi)\mathrm{e}^{-\mathrm{i}(k-l)\omega}\mathrm{d}\omega \\
=\ &\frac{1}{2\pi}\int_{0}^{2\alpha\pi} P^{\beta_1}(z)\tilde{P}^{\beta_1*}(z)\mathrm{e}^{-\mathrm{i}(k-l)\omega}\mathrm{d}\omega \\
=\ &\frac{1}{2\pi}\int_{0}^{2\alpha\pi}\Big(\sum_{j=1}^{\alpha-1} P^{(0)}(\omega_j z)\tilde{P}^{(\lambda)*}(\omega_j z)\Big)\mathrm{e}^{-\mathrm{i}(k-l)\omega}\mathrm{d}\omega = 0.
\end{aligned}$$

Therefore, for any $k,\ l \in \mathbb{Z}$, (33) is established. The proof is completed.

4 The direct decomposition for space $L^2(\mathbb{R})$

In this section, we will decompose subspace V_j, $\widetilde{V}_j$ and W_j, $\widetilde{W}_j$ by virtue of a series of two-direction wavelet packet subspaces. Furthermore, we present the direct decomposition for space $L^2(\mathbb{R})$. Define

$$U_n = \bigvee\Big(\{T^k\phi_n\}_{k\in\mathbb{Z}}\bigcup\{T^l R\phi_n\}_{l\in\mathbb{Z}}\Big),$$

$$\widetilde{U}_n = \bigvee\Big(\{T^k\widetilde{\phi_n}\}_{k\in\mathbb{Z}}\bigcup\{T^l R\widetilde{\phi_n}\}_{l\in\mathbb{Z}}\Big)$$

and define a dilation operator $(S\phi)(t) = \phi(\alpha t)$ where $\phi(t) \in L^2(\mathbb{R})$. We have

$$S^l U_n = U_{\alpha^l n} \oplus U_{\alpha^l n+1} \oplus U_{\alpha^l n+\alpha l-1}, \quad l,\ n \in \mathbb{Z}^+,$$

$$S^l \hat{U}_n = \hat{U}_{\alpha^l n} \oplus \hat{U}_{\alpha^l n+1} \oplus \hat{U}_{\alpha^l n+\alpha l-1}, \quad l,\ n \in \mathbb{Z}^+$$

and $S^0 = 1,\ S^l = S, \cdots, S^{l-1}$. The following result is equivalent to Theorem 3.4.

Lemma 4.1 Let $\{\psi_{\alpha n+\lambda}(t),\ n \in \mathbb{Z}^+,\ \lambda = 0, 1, \cdots, \alpha-1\}$ and $\{\tilde{\psi}_{\alpha n+\lambda}(t),\ n \in \mathbb{Z}^+,\ \lambda = 0, 1, \cdots, \alpha-1\}$ be biorthogonal TDWPs. Then we have the following decomposition formulus:

$$\psi(\alpha t-k) = \frac{1}{\alpha^2}\sum_{j=1}^{\alpha-1}\sum_{l\in\mathbb{Z}}\tilde{P}^{(j)*}_{k-\alpha l}\psi_{\alpha n+j}(t-l), \quad k \in \mathbb{Z}, \tag{34}$$

$$\tilde{\psi}(\alpha t-k) = \frac{1}{\alpha^2}\sum_{j=1}^{\alpha-1}\sum_{l\in\mathbb{Z}}P^{(j)*}_{k-\alpha l}\tilde{\psi}_{\alpha n+j}(t-l), \quad k \in \mathbb{Z}. \tag{35}$$

Proof By the definition of TDWPs and applying (24), we have

$$\begin{aligned}
&\frac{1}{\alpha^2}\sum_{j=1}^{\alpha-1}\sum_{l\in\mathbb{Z}}\tilde{P}^{(j)*}_{k-\alpha l}\psi_{\alpha n+j}(t-l)\\
=&\ \frac{1}{\alpha^2}\sum_{j=1}^{\alpha-1}\sum_{l\in\mathbb{Z}}\tilde{P}^{(j)*}_{k-\alpha l}\sum_{m\in\mathbb{Z}}P^{(j)}_m\psi_n(\alpha t-\alpha l-m)\\
=&\ \frac{1}{\alpha^2}\sum_{j=1}^{\alpha-1}\sum_{l\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\tilde{P}^{(j)*}_{k-\alpha l}P^{(j)}_m\psi_n(\alpha t-\alpha l-m)\\
=&\ \frac{1}{\alpha^2}\sum_{j=1}^{\alpha-1}\sum_{l\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}\tilde{P}^{(j)*}_{k-\alpha l}P^{(j)}_m\psi_{s-\alpha l}(\alpha t-s)\\
=&\ \frac{1}{\alpha^2}\sum_{j=1}^{\alpha-1}\sum_{s\in\mathbb{Z}}\psi_n(\alpha t-s)\sum_{l\in\mathbb{Z}}\tilde{P}^{(j)*}_{k-\alpha l}P^{(j)}_{s-\alpha l}\\
=&\ \Psi_n(\alpha t-k).
\end{aligned}$$

Similarly, one can obtain (35) by using the same method. The proof is completed.

Lemma 4.2 Let $\{\psi_{\alpha n+\lambda}(t),\ n\in\mathbb{Z}^+,\ \lambda=0,1,\cdots,\alpha-1\}$ and $\{\tilde{\psi}_{\alpha n+\lambda}(t),\ n\in\mathbb{Z}^+,\ \lambda=0,1,\cdots,\alpha-1\}$ be biorthogonal TDWPs. Then

$$\big\langle\psi_m(t-k),\ \tilde{\psi}_n(t-l)\big\rangle=\delta_{m,n}\delta_{k,l}I_r,\quad m,\ n=0,1,\cdots,k,\quad l\in\mathbb{Z}. \tag{36}$$

Theorem 4.1 Let $j=0,1,\cdots$, and

$$V_j=\oplus_{0\le n<\alpha^j}U_n,\quad W_j=\oplus_{0\le n<\alpha^j}U_n, \tag{37}$$

$$\widetilde{V}_j=\oplus_{0\le n<\alpha^j}\widetilde{U}_n,\quad \widetilde{W}_j=\oplus_{0\le n<\alpha^j}\widetilde{U}_n, \tag{38}$$

where "$\oplus$" denotes direct sum of subspaces. Then

$$L^2(\mathbb{R})=\oplus_{n\in\mathbb{Z}}U_n=\oplus_{n\in\mathbb{Z}}\tilde{U}_n, \tag{39}$$

where

$$U_m\perp\widetilde{U}_n,\quad m,\ n\in\mathbb{Z}_+,\quad m\neq n, \tag{40}$$

and $\{\psi_{m_\iota}(t-k),\ \psi_{m_\iota}(k-t),\ \iota=1,2,\ k,l\in\mathbb{Z}\}$, $\{\tilde{\psi}_{n_\iota}(t-k),\ \tilde{\psi}_{n_\iota}(k-t),\ \iota=1,2,\ k,l\in\mathbb{Z}\}$ are biorthogonal bases of U_n and $\widetilde{U}_n$, respectively, where $m,\ n\in\mathbb{Z}_+$.

Proof Noting that $V_0=U_0$, $W=\oplus_{\iota=1}^{\alpha}U_\iota$ and $V_j=S^jW_0$, we can obtain (37). Since $\{V_j\}_{j\in\mathbb{Z}}$ and $\{\widetilde{V}_j\}_{j\in\mathbb{Z}}$ are all MRAs of $L^2(\mathbb{R})$, one obtains

$$L^2(\mathbb{R})=V_0\oplus\big(\oplus_{j\in\mathbb{Z}}W_j\big)=\widetilde{V}_0\oplus\big(\oplus_{j\in\mathbb{Z}}\widetilde{W}_j\big).$$

This completes the proof of Theorem 4.1.

5 An Example

In this section, we give an example to demonstrate the general theory in section 3.

Let $\phi(t)$, $\widetilde{\phi}(t)$ be a pair of biorthogonal two direction refinable functions with dilation factor 2. If they satisfy

$$\phi(t) = \frac{1}{2}\phi(2t+1) + \phi(2t) + \frac{1}{2}\phi(2t-1) + \frac{1}{5}\phi(-2t+1) - \frac{1}{5}\phi(-2t-1),$$

$$\widetilde{\phi}(t) = \frac{1}{2}\widetilde{\phi}(2t+1) + \widetilde{\phi}(2t) + \frac{1}{2}\widetilde{\phi}(2t-1) + \frac{5}{4}\widetilde{\phi}(-2t+1) - \frac{5}{4}\widetilde{\phi}(-2t-1),$$

and

$$\phi(-t) = -\frac{2}{5}\phi(-2t+1) + \frac{1}{2}\phi(-2t) - \frac{2}{5}\phi(-2t-1) - \phi(2t+1) + \phi(2t-1),$$

$$\widetilde{\phi}(-t) = -\frac{35}{32}\widetilde{\phi}(-2t+1) + \frac{1}{2}\widetilde{\phi}(-2t) + \frac{35}{32}\widetilde{\phi}(-2t-1) - \frac{7}{16}\widetilde{\phi}(2t+1) + \frac{7}{16}\widetilde{\phi}(2t-1),$$

then $\psi(t)$, $\widetilde{\psi}(t)$ are biorthogonal two direction wavelets associate to $\phi(t)$, $\widetilde{\phi}(t)$, respectively, which satisfy the following equations

$$\psi(t) = \frac{1}{2}\phi(2t+1) - \phi(2t) + \frac{1}{2}\phi(2t-1) + \frac{1}{5}\phi(-2t+1) - \frac{1}{5}\phi(-2t-1),$$

$$\widetilde{\psi}(t) = \frac{1}{2}\widetilde{\phi}(2t+1) - \widetilde{\phi}(2t) + \frac{1}{2}\widetilde{\phi}(2t-1) + \frac{5}{4}\widetilde{\phi}(-2t+1) - \frac{5}{4}\widetilde{\phi}(-2t-1),$$

and

$$\psi(-t) = -\frac{2\sqrt{7}}{35}\phi(-2t+1) - \frac{\sqrt{7}}{2}\phi(-2t) - \frac{2\sqrt{7}}{35}\phi(-2t-1) - \frac{\sqrt{7}}{7}\phi(2t+1) + \frac{\sqrt{7}}{7}\phi(2t-1),$$

$$\widetilde{\psi}(-t) = -\frac{5\sqrt{7}}{32}\widetilde{\phi}(-2t+1) - \frac{\sqrt{7}}{2}\widetilde{\phi}(-2t) - \frac{5\sqrt{7}}{32}\widetilde{\phi}(-2t-1) - \frac{\sqrt{7}}{16}\widetilde{\phi}(2t+1) + \frac{7}{16}\widetilde{\phi}(2t-1).$$

By (30)-(35), we can obtain the biorthogonal two-direction wavelet packets.

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正整数伸缩的双正交双向小波包

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摘　要: 本文引入了尺度为 α 的双正交双向小波包的概念，运用矩阵理论和算子理论研究了双正交双向小波包的性质。得到构造双正交双向小波包的一种新方法。建立了进行迭代与分解的公式。利用双正交双向小波包，得到空间 $L^2(\mathbb{R})$ 新的 Riesz 基。最后，给出构造双正交双向小波包的例子。

关键词: 双向小波；双向小波包；双向加细函数；双正交